[This question paper contains 4 printed pages]

Roll No.

S. No. of Question Paper : 90

Unique Paper Code : 32351301 I

Name of the Paper : Theory of Real Functions

Name of the Course : B.Sc. (Hons.) Mathematics

Semester : III

Duration : 3 Hours Maximum Marks : 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *three* parts from each question.

All questions are compulsory.

(a) Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. Then prove that f can have only one limit at c. 5

(b) Use the $\in$-δ definition of the limit to prove that $\lim\limits_{x \to c} x^3 = c^3$ for any $c \in \mathbb{R}$. 5

(c) State divergence criterion for limit of a function. Show that $\lim\limits_{x \to 0} \left(x + \operatorname{sgn}(x)\right)$ does not exist. 5

P.T.O.

(*d*) Prove that :

(*i*) $\lim_{x \to 0+} \frac{1}{x} = \infty$

(*ii*) $\lim_{x \to \infty} \frac{1}{x^2} = 0.$

2. (*a*) Let $A \subseteq R$, $f, g, h : A \to R$ and $c \in R$ be a clust point of A. If $f(x) \le g(x) \le h(x)$ for all $x \in A$, $x \neq$ and if $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, then prove th $\lim_{x \to c} g(x) = L$.

(*b*) State and prove sequential criterion for continuity of real valued function.

(*c*) Let the function $f : R \to R$ be defined by

$$f(x) = \begin{cases} 2x & : \text{ if } x \text{ is rational} \\ x+3 & : \text{ if } x \text{ is irrational} \end{cases}$$

Find all the points at which f is continuous.

(*d*) Let $x \to [x]$ denote the greatest integer function. Determ the points of continuity of the function $f(x) = x - [$ $x \in R$.

3. (*a*) Let f be a continuous real valued function defined on $[a, b]$. By assuming that f is a bounded function show that f attains its bounds on $[a, b]$. 5

(*b*) State Bolzano's Intermediate value theorem and show that the function $f(x) = xe^x - 2$ has a root c in the interval $[0, 1]$. 5

(*c*) Let $f : R \to R$ is continuous on R and suppose that $f(r) = 0$ for every rational numbers r. Show that $f(x) = 0$ for all $x \in R$. 5

(*d*) Define uniform continuity of a function. Prove that if a function is continuous on a closed and bounded interval I, then it is uniformly continuous on I. 5

4. (*a*) Show that the function $f(x) = 1/x^2$ is uniformly continuous on $A = [0, \infty[$ but it is not uniformly continuous on $B =]0, \infty[$. 5

(*b*) Determine where the following function $f : R \to R$ is differentiable, $f(x) = |x - 1| + |x + 1|$. 5

P.T.O.

(c) Let f be defined on an interval I containing the point c. Then prove that f is differentiable at c if and only if there exists a function ϕ on I that is continuous at c and satisfies $f(x) - f(c) = \phi(c)(x - c)$ for all $x \in$ I. In this case, we have $\phi(c) = f'(c)$. Using the above result find the function ϕ for $f(x) = x^3$, $x \in$ R.

(d) State and prove Mean Value Theorem.

5. (a) State Darboux's theorem. Suppose that $f : [0, 2] \to$ R is continuous on [0, 2] and differentiable on]0, 2[and that $f(0) = 0$, $f(1) = 1$, $f(2) = 1$. (i) Show that there exist $c_1 \in (0, 1)$ such that $f'(c_1) = 1$. (ii) Show that there exist $c_2 \in (1, 2)$ such that $f'(c_2) = 0$. (iii) Show that there exist $c \in (0, 2)$ such that $f'(c) = 1/10$.

(b) Let $f : I \to$ R be differentiable on the interval I. Then prove that f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in$ I.

(c) State Taylor's theorem. Use it to prove that $1 - x^2/2 \leq \cos x$ for all $x \in$ R.

(d) Find the Taylor series for e^x and state why it converges to e^x for all $x \in$ R.